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Perfect sets of finite class without the extension property

by

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Abstract. We prove that generalized Cantor sets of class $\alpha, \alpha \neq 2$, have the extension property iff $\alpha < 2$. Thus belonging of a compact set K to some finite class α cannot be a characterization for the existence of an extension operator. The result has some interconnection with potential theory.

1. Introduction. Let K be a compact set in $\mathbb{R}^m$. Then $\mathcal{E}(K)$ is the space of Whitney jets with the topology defined by the norms (in what follows we will consider only the one-dimensional case)

$$\|f\|_q = |f|_q + \sup\left\{\frac{|(R_y^q f)^{(k)}(x)|}{|x-y|^{q-k}} : x, y \in K,\ x \neq y,\ k = 0, 1, \ldots, q\right\},$$

$q = 0, 1, \ldots,$ where $|f|_q = \sup\{|f^{(k)}(x)| : x \in K,\ k \le q\}$ and $R_y^q f(x) = f(x) - T_y^q f(x)$ is the Taylor remainder. We say that K *has the extension property* if there exists a linear continuous extension operator $L : \mathcal{E}(K) \to C^\infty(\mathbb{R}^m)$. The problem of finding such an operator was investigated by many authors (see e.g. [2], [9], [11], [12], [14]–[17]). In [16] Tidten applied Vogt's condition for a splitting of exact sequences of Fréchet spaces and gave a topological characterization of the extension property (see Th. 1 below). In order to give a corresponding geometric description Tidten introduced in [17] the following property: a compact set $K \subset \mathbb{R}$ is a *perfect set of class* α ($\alpha \ge 1$) if there are constants $C \ge 1$ and $\delta > 0$ such that for any $y \in K$ one can find a sequence $(x_j)_{j=1}^\infty \subset K$ such that $|y - x_j| \downarrow 0, |y - x_1| \ge \delta$ and $C|y - x_{j+1}| \ge |y - x_j|^\alpha$ for any $j \in \mathbb{N}$. In this case we will write $K \in (\alpha)$. It was proved in [17] that

(i) $K \in (1) \Rightarrow$
(ii) K has the extension property $\Rightarrow$
(iii) $K \in (\alpha)$ for some $\alpha \ge 1$.

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If K has the form of a sequence of closed intervals tending to a point, then (under a minor restriction of regularity) the conditions (ii) and (iii) are equivalent ([4]).

Nevertheless the class (α) cannot be in general a characterization of the extension property. We give here examples of generalized Cantor sets of finite class (α) without (ii). Some interconnection of the extension property and potential theory is presented for our case.

We shall use the class D_1 (see [19]) or the property DN (see [18]) of Fréchet spaces:

$$\exists p\ \forall q\ \exists r, C>0: \quad \|\cdot\|_q \le t\|\cdot\|_p + \frac{C}{t}\|\cdot\|_r, \qquad t>0. \tag{1}$$

Here and in the sequel we consider (F) spaces with an increasing system of seminorms; $p,q,r\in\mathbb{N}=\{0,1,\ldots\}$.

THEOREM 1 (Tidten [16], Folg. 2.4). *A compact set K has the extension property iff the space $\mathcal{E}(K)$ has the property DN.*

PROPOSITION 1. *The following statements are equivalent to DN:*

$$\exists p\ \exists R>0\ \forall q\ \exists r, C: \quad \|\cdot\|_q \le t^{Rq}\|\cdot\|_p + \frac{C}{t^q}\|\cdot\|_r, \qquad t>0; \tag{2}$$

$$\exists p\ \forall \varepsilon>0\ \forall q\ \exists r, C: \quad \|\cdot\|_q^{1+\varepsilon} \le C\|\cdot\|_p\cdot\|\cdot\|_r^{\varepsilon}. \tag{3}$$

Proof. For the equivalence (1)⇔(2) see e.g. [3]; (1)⇔(3) can be found in [8], Lemma 29.10. ∎

2. Cantor type sets without the extension property. Let $(l_n)_{n=0}^{\infty}$ be a sequence such that $l_0=1$, $0<2l_{n+1}<l_n$, $n\in\mathbb{N}$. Let K be the Cantor set associated with the sequence (l_n), that is, $K=\bigcap_{n=0}^{\infty}K_n$, where $K_0=I_{0,1}=[0,1]$, K_n is a union of 2^n closed intervals $I_{n,k}$ of length l_n and K_{n+1} is obtained by deleting the open concentric subinterval of length l_n-2l_{n+1} from each $I_{n,k}$, $k=1,2,\ldots,2^n$.

Fix $\alpha>1$ and $l_1<1/2$ with $2l_1^{\alpha-1}<1$. We will denote by $K^{(\alpha)}$ the Cantor set associated with the sequence (l_n), where $l_0=1$, $l_{n+1}=l_n^{\alpha}=\ldots=l_1^{\alpha^n}$, $n\ge1$. From the definition of the class (α) we have (see also [15], Prop. 2.1)

PROPOSITION 2. *$K^{(\alpha)}\in(\alpha)$ and $K^{(\alpha)}\notin(\beta)$, $\forall\beta<\alpha$.*

Our purpose is to show that for $\alpha>2$ the space $\mathcal{E}(K^{(\alpha)})$ does not satisfy (3). First we give a sharpened version of Lemma 3 from [4].

LEMMA 1. *Let $g(x)=\prod_{j=1}^{N}(x-a_j)$, where $|x-a_j|\le l<1$, $j=1,\ldots,N$. Let $f(x)=g^q(x)$. Then for $n\le Nq$,*

$$|f^{(n)}(x)|\le C(N,q,n)l^{Nq-n}. \tag{4}$$

If in addition $n<q$, then

$$|f^{(n)}(x)|\le C(N,q,n)|g(x)|^{q-n}. \tag{5}$$

Here

$$C(N,q,n)=\frac{(Nq)!}{(Nq-n)!}.$$

Proof. By the Faa di Bruno formula for the derivative of superposition (see e.g. [5], 0.430) we have

$$f^{(n)}(x)=\sum\frac{n!}{k_1!\ldots k_n!}\cdot\frac{q!}{(q-k)!}g^{q-k}(x)\prod_{i=1}^{n}\left(\frac{g^{(i)}(x)}{i!}\right)^{k_i}. \tag{6}$$

Here the sum is taken over all sequences $(k_1,\ldots,k_n)\in\mathbb{N}^n$ such that $k_1+2k_2+\ldots+nk_n=n$ and $k:=k_1+\ldots+k_n\le q$. In the case $N<n$ all terms corresponding to $(k_1,\ldots,k_n)$ with $k_i\ne0$ for some $i>N$ vanish. If $i\le\nu:=\min\{N,n\}$, then $g^{(i)}(x)$ is a sum of $N!/(N-i)!$ terms and every term is a product of $N-i$ factors of type $x-a_j$. Therefore

$$|f^{(n)}(x)|\le\sum\frac{n!}{k_1!\ldots k_n!}\cdot\frac{q!}{(q-k)!}\prod_{i=1}^{\nu}\binom{N}{i}^{k_i}l^{\sigma}, \tag{7}$$

where $\sigma=N(q-k)+\sum_{i=1}^{\nu}(N-i)k_i$. If $n\le N$, then $\sigma=N(q-k)+Nk-n=Nq-n$. If $n>N$, then

$$\sigma=Nq-N\sum_{i=N+1}^{n}k_i-n+\sum_{i=N+1}^{n}ik_i\ge Nq-n.$$

Thus, $|f^{(n)}(x)|\le C(N,q,n)l^{Nq-n}$, where the coefficient is the right side of (7) without l^{σ}. In order to find it one can take $g=x^N$ and apply (6) at $x=1$: $f^{(n)}(1)=C(N,q,n)$. On the other hand, $f^{(n)}(1)=(Nq)!/(Nq-n)!$.

If $n<q$, then we neglect all factors of type $l^{(N-i)k_i}$. Since $k\le n$ and $|g|^{q-k}\le|g|^{q-n}$, we get (5). ∎

THEOREM 2. *If $\alpha>2$, then $K^{(\alpha)}$ does not have the extension property.*

Proof. Fix $\alpha>2$, $\varepsilon=(\alpha-2)/2$ and $M\in\mathbb{N}$ such that $M\ge2\alpha/(\alpha-2)$. We will show the negation of (3):

$$\forall p\ \exists\varepsilon\ \exists q\ \forall r>q\ \exists(f_n)\subset\mathcal{E}(K^{(\alpha)}):\quad\frac{\|f_n\|_p\|f_n\|_r^{\varepsilon}}{\|f_n\|_q^{1+\varepsilon}}\to0\quad\text{as } n\to\infty.$$

For arbitrary $p\in\mathbb{N}$ let $q=Mp+1$. For any $r>q$ take $s\in\mathbb{N}$ with $2^s\ge r/q>2^{s-1}$. Fix natural $n\ge s+3$, and consider the first 2^s intervals of K_n: $I_{n,1}=[0,l_n]$, $I_{n,2}=[l_{n-1}-l_n,l_{n-1}],\ldots,I_{n,2^s}=[l_{n-s}-l_n,l_{n-s}]$. Let c_j denote the midpoint of $I_{n,j}$, $j=1,\ldots,2^s$. Set $f_n(x)=g^q(x)$ where $g(x)=\prod_{j=1}^{2^s}(x-c_j)$ for $x\in K^{(\alpha)}\cap[0,l_{n-s}]$ and $g(x)=0$ elsewhere on $K^{(\alpha)}$. Let us evaluate the norms of f_n.

Upper bound of $\|f_n\|_p$. Fix natural $k \le p$ and $x \in \bigcup_{j=1}^{2^s} I_{n,j}$. By (5) we have

$$|f_n^{(k)}(x)| \le C(2^s, q, k)|g(x)|^{q-k}.$$

Throughout the proof let C_p denote $C(2^s, q, p) = \max_{k \le p} C(2^s, q, k)$. It follows from the structure of the set $K^{(\alpha)}$ that $|g(x)|$ is a product of 2^s terms where one term is less than l_n, another is less than l_{n-1}, two others are less than $l_{n-2}, \ldots, 2^{s-1}$ largest terms are less than l_{n-s}. Therefore, $|g(x)| < l_n\lambda$, where λ denotes (here and in the sequel) $l_{n-1}\, l_{n-2}^2 \ldots l_{n-s}^{2^{s-1}}$. Thus,

$$|f_n^{(k)}(x)| \le C_p(l_n\lambda)^{q-k}. \tag{8}$$

From this $|f|_p \le C_p(l_n\lambda)^{q-k}$. Furthermore, we can estimate

$$A_p := \frac{|(R_x^p f_n)^{(k)}(y)|}{|x-y|^{p-k}}, \qquad k \le p,\ x \ne y,\ x, y \in K^{(\alpha)}.$$

If $|x-y| < l_{n-1} - 2l_n$, then x, y belong to the same interval $I_{n,j}$ for some j. Applying the Lagrangian form for Taylor's remainder we find $\xi \in I_{n,j}$ such that

$$(R_x^p f_n)^{(k)}(y) = [f^{(p)}(\xi) - f^{(p)}(x)] \cdot \frac{(y-x)^{p-k}}{(p-k)!}.$$

Therefore, $A_p \le 2C_p(l_n\lambda)^{q-p}$.

Let $|x-y| \ge l_{n-1} - 2l_n = l_{n-1}(1 - 2l_{n-1}^{\alpha-1})$. Since $\alpha > 2$ and $n \ge s+3 \ge 4$, we see that $l_{n-1}^{\alpha-1} < l_{n-1} \le l_2 < 1/4$. Then $|x-y| \ge l_{n-1}/2$ and by (8),

$$\begin{aligned} A_p &\le |f_n^{(k)}(y)| \cdot |x-y|^{k-p} + \sum_{i=k}^{p} |f_n^{(i)}(x)| \frac{|x-y|^{i-p}}{(i-k)!} \\ &\le C_p(l_n\lambda)^{q-p}\left[\left(2 \cdot \frac{l_n\lambda}{l_{n-1}}\right)^{p-k} + \sum_{i=k}^{p} \frac{1}{(i-k)!}\left(2 \cdot \frac{l_n\lambda}{l_{n-1}}\right)^{p-i}\right]. \end{aligned}$$

Since $2l_n\lambda < l_{n-1}$, we get the uniform bound $A_p \le C_p(1+e)(l_n\lambda)^{q-p}$. Therefore,

$$\|f_n\|_p \le (2+e)C_p(l_n\lambda)^{q-p}.$$

Lower bound of $\|f_n\|_q$. Clearly, $\|f_n\|_q \ge |f_n|_q \ge |f_n^{(q)}(c_1)|$. If we apply (6) for $n = q$ and $x = c_1$, we see that the only nonzero term in the sum corresponds to the case $k_1 = q$, $k_2 = \ldots = k_q = 0$. From this,

$$f_n^{(q)}(c_1) = q!(g'(c_1))^q.$$

Here $|g'(c_1)| = \prod_{j=2}^{2^s}(c_j - c_1)$ and $c_2 - c_1 = l_{n-1} - l_n > l_{n-1}/2$; $c_3 - c_1 = l_{n-2} - l_{n-1} > l_{n-2}/2$; $c_4 - c_1 > c_3 - c_1 > l_{n-2}/2; \ldots; c_{2^s} - c_1 > l_{n-s} - l_{n-s+1} > l_{n-s}/2$. Therefore,

$$|g'(c_1)| > \frac{l_{n-1}}{2}\left(\frac{l_{n-2}}{2}\right)^2 \ldots \left(\frac{l_{n-s}}{2}\right)^{2^{s-1}} = \frac{\lambda}{C},$$

where $C = 2^{2^s-1} < 2^{2r/q}$. Finally, we get

$$\|f_n\|_q \ge q!2^{-2r}\lambda^q.$$

Upper bound of $\|f_n\|_r$. Let $k \le r$ and $x \in [0, l_{n-s}]$. Since $2^s q \ge r$ from (4) we conclude that

$$|f^{(k)}(x)| \le C(2^s, q, k)l_{n-s}^{r-k}.$$

Therefore, $|f|_r \le C_r := \max_{k \le r} C(2^s, q, k)$.

Let now $A_r = |(R_x^r f)^{(k)}(y)| \cdot |x-y|^{k-r}$, $k \le r$, $x, y \in K^{(\alpha)}$. If $x, y \in [0, l_{n-s}]$, then arguing as above we see that for some point $\xi \in (0, l_{n-s})$,

$$A_r \le |f^{(r)}(\xi) - f^{(r)}(x)| \le 2C_r.$$

Otherwise, $|x-y| \ge l_{n-s-1} - 2l_{n-s} > l_{n-s}$ as $n \ge s+3$. (We exclude the trivial case: $x, y \notin \operatorname{supp} g(x)$.) If $x \ge l_{n-s-1} - l_{n-s}$ and $y \le l_{n-s}$, then

$$A_r = |f^{(k)}(y)| \cdot |x-y|^{k-r} \le C_r l_{n-s}^{r-k}|x-y|^{k-r} < C_r.$$

If $x \le l_{n-s}$ and $y \ge l_{n-s-1} - l_{n-s}$, then

$$A_r \le \sum_{i=k}^{r} |f^{(i)}(x)| \frac{|x-y|^{i-r}}{(i-k)!} \le C_r e.$$

Thus,

$$\|f_n\|_r \le C_r(1+e).$$

Now we can estimate the corresponding fraction:

$$\frac{\|f_n\|_p \|f_n\|_r^{\varepsilon}}{\|f_n\|_q^{1+\varepsilon}} \le \widetilde{C}\frac{(l_n\lambda)^{q-p}}{\lambda^{q(1+\varepsilon)}} = \widetilde{C} l_n^{q-p}\lambda^{-q\varepsilon-p}, \tag{9}$$

where the constant $\widetilde{C}$ does not depend on n.

By the definition, $l_{n-k} = l_{n-s}^{\alpha^{s-k}}$, $k = 0, 1, \ldots, s$, and

$$\lambda = l_{n-s}^{\alpha^{s-1}} \cdot l_{n-s}^{2\cdot\alpha^{s-2}} \ldots l_{n-s}^{2^{s-1}} = l_{n-s}^{\omega},$$

where

$$\omega = \sum_{k=1}^{s} 2^{k-1}\alpha^{s-k} = \frac{\alpha^s - 2^s}{\alpha - 2} < \frac{\alpha^s}{\alpha - 2}.$$

Therefore the right side of (9) is equal to $\widetilde{C} l_{n-s}^{(q-p)\alpha^s - (q\varepsilon+p)\omega}$. Let us show that the exponent of l_{n-s} here is positive. Then the right side of (9) tends to 0 as $n \to \infty$, which completes the proof.

In fact,

$$(q-p)\alpha^s - (q\varepsilon + p)\omega > (q-p)\alpha^s - \left(q\frac{\alpha-2}{2} + p\right)\frac{\alpha^s}{\alpha-2}$$
$$= q\frac{\alpha^s}{2} - p\alpha^s\frac{\alpha-1}{\alpha-2} > p\alpha^s\left(\frac{M}{2} - \frac{\alpha-1}{\alpha-2}\right) > 0,$$

due to the choice of M. ■

3. Cantor type sets with the extension property. In [17] Tidten has shown that the Cantor set has the extension property as a perfect set of class (1). Let us extend this result to the case $1 < \alpha < 2$. First we give a general form of Lemma 2 from [4].

For $r+1$ distinct points $(x_i)_{i=0}^r$ let $h_i = |x_i - x_0|$, $i = 1, \ldots, r$; let $\pi(x)$ denote the polynomial $\prod_{i=0}^r (x - x_i)$; $\mathcal{E}^r(K)$ is the Banach space of r times differentiable Whitney jets on K equipped with the norm $\|\cdot\|_r$.

LEMMA 2. *Let K be a compact set containing $r+1$ points $(x_i)_{i=0}^r$ such that $h_i \le h_{i+1}$, $i = 1, \ldots, r-1$. Then for any $f \in \mathcal{E}^r(K)$ and $1 \le k \le r$,*

$$|f^{(k)}(x_0)| \le 2C|f|_0\mu_1 + C\|f\|_r\mu_2,$$

where

$$C = \frac{r!k}{(r-k)!}, \quad \mu_1 = h_{k+1}\ldots h_r \max_{1\le i\le r}\frac{1}{|\pi'(x_i)|},$$
$$\mu_2 = h_{k+1}\ldots h_r \max_{1\le i\le r}\frac{h_i^r}{|\pi'(x_i)|}.$$

Proof. Fix $f \in \mathcal{E}^r(K)$. Let $F_i = f(x_i) - f(x_0) - R_{x_0}^r f(x_i)$, $i = 1, \ldots, r$. Consider the system of equations

$$\sum_{k=1}^r \frac{f^{(k)}(x_0)}{k!}(x_i - x_0)^k = F_i, \quad i = 1, \ldots, r,$$

with the "unknowns" $f^{(k)}(x_0)/k!$, $k = 1, \ldots, r$. The coefficients of the system give the Vandermonde determinant $V = V(x_0, x_1, \ldots, x_r) = \prod_{i<j}(x_j - x_i)$. Applying the symmetric functions $S_0 = 1$, $S_j(a_1, \ldots, a_n) = a_1 a_2 \ldots a_j + \ldots + a_{n-j+1}\ldots a_n$ (the sum of $\binom{n}{j}$ products of j factors without repetition), we have the following expression of the auxiliary determinant $\triangle_k$, $k = 1, \ldots, r$ (see [4] for more details):

$$\triangle_k = (-1)^{r+k}\sum_{i=1}^r F_i \frac{V}{\pi'(x_i)} S_{r-k}(x_1 - x_0, \ldots, x_{i-1} - x_0, x_{i+1} - x_0, \ldots, x_r - x_0).$$

By Cramer's rule, omitting the argument of the symmetric function, we get

$$\frac{f^{(k)}(x_0)}{k!} = (-1)^{r+k}\sum_{i=1}^r F_i \frac{S_{r-k}}{\pi'(x_i)}, \quad k = 1, \ldots, r.$$

Here $|S_{r-k}| \le \binom{r-1}{r-k} h_{k+1}h_{k+2}\ldots h_r$ and $|F_i| \le 2|f|_0 + \|f\|_r h_i^r$, which proves the lemma. ■

THEOREM 3. *If $1 < \alpha < 2$, then $K^{(\alpha)}$ has the extension property.*

Proof. Let us show (2) for the space $\mathcal{E}(K^{(\alpha)})$. Given $\alpha \in (1,2)$ let

$$C_\alpha = \left(\frac{4}{2-\alpha}\right)^{\frac{1}{1-\log_2\alpha}}, \quad R = \frac{24C_\alpha}{2-\alpha}.$$

Take $p = 0$ and arbitrary natural $q \ge 1$. For $v = \min\{k \in \mathbb{N} : 2^k - 1 \ge 2q\}$ let $q_1 = 2^v - 1$; then $2q \le q_1 \le 4q$. Fix natural s such that $r := 2^s - 1 \ge C_\alpha q_1 > 2^{s-1} - 1$. Then $r/q_1 < 3C_\alpha$. Fix $f \in \mathcal{E}(K^{(\alpha)})$ and $t > 4^{1/\alpha - 1}$. Let n be such that

$$l_{n-s} < 1/t \le l_{n-s-1} = l_{n-s}^{1/\alpha}. \tag{10}$$

We first estimate $|f^{(k)}(x_0)|t^{q_1-k}$, $x_0 \in K^{(\alpha)}$, $k \le q_1$. To this end, consider $K_n = \bigcup_{j=1}^{2^n} I_{n,j} \supset K^{(\alpha)}$. Let $x_0 \in I_{n,j_0}$. Also, $x_0 \in I_{n-s,j_1} \subset K_{n-s}$. The interval I_{n-s,j_1} covers 2^s intervals of K_n. Let us take the right endpoints of these intervals except I_{n,j_0} and enumerate them in the order of increasing distance to x_0. Thus we have $r+1$ distinct points $(x_i)_{i=0}^r$ in $K^{(\alpha)}$. Clearly, $h_r = |x_r - x_0| \le l_{n-s}$. In order to use Lemma 2 let us bound μ_1, μ_2 in our case. On the one hand, $|\pi'(x_i)|$ is a product of $2^s - 1$ terms where one term is more than $l_{n-1} - 2l_n$, two others are more than $l_{n-2} - 2l_{n-1}, \ldots, 2^{s-1}$ terms are more than $l_{n-s} - 2l_{n-s+1}$. From (10) and by the choice of t we see that $4l_{n-s}^{\alpha-1} < 1$, hence $l_{n-s} - 2l_{n-s+1} > l_{n-s}/2$. All the more for $i \le s-1$ we get $l_{n-i} - 2l_{n-i+1} > l_{n-i}/2$. Therefore,

$$|\pi'(x_i)| \ge \frac{l_{n-1}}{2}\left(\frac{l_{n-2}}{2}\right)^2 \ldots \left(\frac{l_{n-s}}{2}\right)^{2^{s-1}} = \frac{\lambda}{2^r}.$$

On the other hand, arguing as above, one can show that

$$|\pi'(x_0)| = h_1 \ldots h_r < \lambda.$$

For this reason

$$\mu_1 = (h_1 \ldots h_k)^{-1}\max\left|\frac{\pi'(x_0)}{\pi'(x_1)}\right| < \frac{2^r}{h_1\ldots h_k}, \quad \mu_2 < \frac{2^r h_r^r}{h_1 \ldots h_k}.$$

Also,

$$h_1 \ldots h_{q_1} \ge \frac{l_{n-1}}{2}\ldots\left(\frac{l_{n-v}}{2}\right)^{2^{v-1}} = 2^{-q_1} l_{n-s}^{\chi}, \tag{11}$$

where

$$\chi = \sum_{i=1}^{v} 2^{i-1}\alpha^{s-i} = \alpha^{s-v}\frac{2^v - \alpha^v}{2-\alpha}.$$

Let us show that

(12) $$\alpha\chi \le Rq \quad \text{and} \quad \chi + q \le r.$$

In fact,

$$\alpha^{s-v} = \left(\frac{r+1}{q_1+1}\right)^{\log_2\alpha} < \left(\frac{r}{q_1}\right)^{\log_2\alpha} < 3C_\alpha^{\log_2\alpha}.$$

Since $2^v - \alpha^v < q_1 \le 4q$, we get $\alpha\chi < 2\chi < Rq$. Moreover,

$$\chi + q < 3C_\alpha^{\log_2\alpha}\frac{q_1}{2-\alpha} + \frac{q_1}{2} < \frac{4}{2-\alpha}C_\alpha^{\log_2\alpha}q_1 = C_\alpha q_1 \le r.$$

Combining (10)–(12), we get

$$\mu_1 < \frac{2^r}{h_1\dots h_{q_1}}l_{n-s}^{q_1-k} \le 2^{r+q_1}l_{n-s}^{-\chi}l_{n-s}^{q_1-k} \le 2^{r+q_1}t^{Rq}t^{-q_1+k},$$

$$\mu_2 < 2^{r+q_1}l_{n-s}^{q_1-k+r-\chi} \le 2^{r+q_1}t^{-q_1+k-q}.$$

Now by Lemma 2 we have

(13) $$|f^{(k)}(x_0)|t^{q_1-k} \le 2^{r+q_1}C(2|f|_0t^{Rq} + \|f\|_r t^{-q}), \qquad \forall x_0 \in K^{(\alpha)},\ k \le q_1.$$

To shorten notation, we write $S(t)$ for the right side of (13). We see that $|f|_q \le S(t)$. It remains to estimate $A_q = |(R_x^q f)^{(k)}(y)|\cdot|x-y|^{k-q}$, $k \le q$, $x, y \in K^{(\alpha)}$.

If $|x-y| \ge 1/t$, then

$$A_q \le |f^{(k)}(y)|\cdot|x-y|^{k-q} + \sum_{i=k}^{q}|f^{(i)}(x)|\frac{|x-y|^{i-q}}{(i-k)!} \le |f^{(k)}(y)|t^{q-k} + \sum_{i=k}^{q}|f^{(i)}(x)|\frac{t^{q-i}}{(i-k)!}$$

$$\le S(t)t^{-q}(1+e), \qquad \text{by (13)}.$$

If $|x-y| < 1/t$, then $R_x^q f(y) = R_x^{q_1}f(y) + \sum_{i=q+1}^{q_1} f^{(i)}(x)(y-x)^i/i!$. Hence

$$A_q \le \|f\|_{q_1}\cdot|x-y|^{q_1-k} + \sum_{i=q+1}^{q_1}|f^{(i)}(x)|\frac{|x-y|^{i-q}}{(i-k)!} \le \|f\|_r t^{-q} + S(t)t^{q-q_1}e,$$

as $q \le q_1 - k$ with $k \le q$.

Thus, $A_q \le 2S(t)$ and

$$\|f\|_q < 3S(t) = C_1|f|_0t^{Rq} + C_2\|f\|_r t^{-q}$$

with $t^{\alpha-1} > 4$, where C_1, C_2 do not depend on f and t. This easily implies (2), which completes the proof. ∎

It is interesting to note that for the Cantor sets $K^{(\alpha)}$ the value $\alpha = 2$ is also a limiting value in potential theory. It is easy to see that the function

$$\varphi_\alpha(\tau) = \left(\ln\frac{1}{\tau}\right)^{-\frac{\ln 2}{\ln\alpha}}$$

is associated (see [10], V, 6.7) with the set $K^{(\alpha)}$.

COROLLARY 1. *For the Cantor set $K^{(\alpha)}$, $1 < \alpha$, $\alpha \ne 2$, the following statements are equivalent*:

(i) $\alpha < 2$;
(ii) *the (logarithmic) capacity of $K^{(\alpha)}$ is positive*;
(iii) *the logarithmic measure of $K^{(\alpha)}$ is positive (or infinite)*;
(iv) *$K^{(\alpha)}$ is regular in the sense of the Green function of $\mathbb{C}\setminus K^{(\alpha)}$ with a pole at ∞*;
(v) *$K^{(\alpha)}$ has the extension property.*

Proof. The equivalence (i)⇔(ii) follows e.g. from Theorem 3 of [10]; and (i)⇔(iii) e.g. from Theorem 4 of [10] and 4.5.2 of [6] as $\varphi_\alpha(\tau) = o(h(\tau))$ for $\alpha < 2$ and $h(\tau) = o(\varphi_\alpha(\tau))$ for $\alpha > 2$, where $h(\tau) = (\ln(1/\tau))^{-1}$; for (ii)⇔(iv) see Proposition 2 of [13]; (i)⇔(v) is the content of the present paper. ∎

EXAMPLE. Let $K = \{0\}\cup\bigcup_{n=2}^{\infty} I_n$, where $I_n = [1/n + \psi_n]$ with $\psi_n \le 1/n^2$. Let $\gamma_n = -(\ln\psi_n)/\ln n$. It follows from Theorem 3 of [4] that $\mathcal{E}(K)$ has the extension property iff the sequence (γ_n) is bounded. On the other hand, by Wiener's criterion (see e.g. [7], Theorem 5.6) the compact set K is regular iff $\sum 1/\gamma_n$ diverges. Therefore the case $\psi_n = n^{-n}$ gives us a regular compact set; $\psi_n = n^{-n^2}$ gives an irregular one at $x = 0$. Neither has the extension property.

Remark. In view of Pleśniak's result ([13], Prop. 1) (see also [1]) for any $\alpha > 1$ the Markov inequality is not satisfied for some polynomials on $K^{(\alpha)}$, but in the case $1 < \alpha < 2$ the compact set $K^{(\alpha)}$ preserves the extension property (compare this with [3]).

QUESTION. What is a geometric characterization of the extension property?

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On the range of convolution operators on non-quasianalytic ultradifferentiable functions

by

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Abstract. Let $\mathcal{E}_{(\omega)}(\Omega)$ denote the non-quasianalytic class of Beurling type on an open set Ω in $\mathbb{R}^n$. For $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ the surjectivity of the convolution operator $T_\mu : \mathcal{E}_{(\omega)}(\Omega_1) \to \mathcal{E}_{(\omega)}(\Omega_2)$ is characterized by various conditions, e.g. in terms of a convexity property of the pair (Ω_1, Ω_2) and the existence of a fundamental solution for μ or equivalently by a slowly decreasing condition for the Fourier–Laplace transform of μ. Similar conditions characterize the surjectivity of a convolution operator $S_\mu : \mathcal{D}'_{\{\omega\}}(\Omega_1) \to \mathcal{D}'_{\{\omega\}}(\Omega_2)$ between ultradistributions of Roumieu type whenever $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^n)$. These results extend classical work of Hörmander on convolution operators between spaces of C^∞-functions and more recent one of Ciorănescu and Braun, Meise and Vogt.

Since the classical work of Ehrenpreis [10] and Hörmander [14], convolution operators on various spaces of infinitely differentiable functions and distributions have been investigated by many authors (see e.g. Berenstein and Dostal [1], Chou [8], Ciorănescu [9], Franken and Meise [11], v. Grudzinski [12], Meise, Taylor and Vogt [20], Braun, Meise and Vogt [7], Meyer [23], Momm [24], [25]). The starting point for the research presented here was a recent result of Bonet and Galbis [3]. They proved that each convolution operator T_μ acting on the non-quasianalytic class $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ (defined in the sense of Braun, Meise and Taylor [6]) for which $T_\mu(\mathcal{E}_{(\omega)}(\mathbb{R}^n))$ contains some smaller class $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ already acts surjectively on $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$.

In the present paper we show that this holds in greater generality and is an immediate corollary to the following extension of results of Hörmander [14] to the non-quasianalytic classes $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ (see 2.7–2.9).

THEOREM A. *Let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^n)$ and open sets Ω_1, Ω_2 in $\mathbb{R}^n$ with $\Omega_1 + \operatorname{Supp} \check{\mu} \subset \Omega_2$ be given. Then the following conditions are equivalent*:

(1) *For each $g \in \mathcal{E}_{(\omega)}(\Omega_1)$ there exists $f \in \mathcal{E}_{(\omega)}(\Omega_2)$ with $\mu * f|_{\Omega_1} = g$.*
(2) *For each $g \in \mathcal{E}_{(\omega)}(\Omega_1)$ there exists $f \in \mathcal{D}'_{(\omega)}(\Omega_2)$ with $\mu * f|_{\Omega_1} = g$.*

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